Today

Philips Saturn 563 AM Radio

1. More properties of the Fourier transform
2. Parseval's identity
3. LTI systems and the transfer function
4. Modulation
5. Sampling
**Theorem**

Let \( x(t) \) and \( y(t) \) be piecewise continuous, absolutely integrable and bounded signals, with Fourier transforms \( X(\omega) \) and \( Y(\omega) \) respectively. Then the \( \mathcal{F}\{x \ast y\} \) exists and

\[
\mathcal{F}\{x \ast y\} = X(\omega) \cdot Y(\omega).
\]

**Equation 4.3.20**

**Example**

Find the Fourier transform of the triangle function

\[
x(t) = \begin{cases} 
a + t & \text{if } -a < t < 0, \\
a - t & \text{if } 0 < t < a, \\
0 & \text{otherwise.}
\end{cases}
\]

- See exercise 2.3(b): the triangle function is a convolution:

\[
x(t) = \text{rect}(t/a) \ast \text{rect}(t/a).
\]

- \( \text{rect}(t/a) \leftrightarrow a \text{Sa} \left( \frac{a \omega}{2} \right) \)

- \( x(t) \leftrightarrow a^2 \text{Sa}^2 \left( \frac{a \omega}{2} \right) = \frac{4}{\omega^2} \sin^2 \left( \frac{a \omega}{2} \right) = \frac{1 - \cos(2\omega)}{2\omega^2}. \)
For an LTI system the response is completely determined by the impulse response $h$. If the input is $x(t)$ then

$$y(t) = (x * h)(t)$$

Define the transfer function $H(\omega) = \mathcal{F}\{h(t)\}$, then

$$Y(\omega) = X(\omega)H(\omega)$$

Example: consider the integrator $x(t) \mapsto \int_{-\infty}^{t} x(\tau) \, d\tau$ with impulse response $u(t)$, then

$$Y(\omega) = X(\omega)U(\omega)$$

$$= X(\omega) \left( \pi \delta(\omega) + \frac{1}{i\omega} \right)$$

$$= \pi X(0) \delta(\omega) + \frac{X(\omega)}{i\omega}. \quad \text{eq. 4.3.11}$$

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**Differentiation in the time domain**

**Theorem**

Lets $x(t)$ be a piecewise smooth signal with Fourier transform $X(\omega)$ and assume that $\lim_{t \to \infty} x(t) = 0$ and $\lim_{t \to -\infty} x(t) = 0.$

Then $\mathcal{F}\{x'(t)\}$ exists and

$$\mathcal{F}\{x'(t)\} = i\omega X(\omega).$$

Use integration by parts to prove the theorem:

$$\int_{M}^{L} x'(t)e^{-i\omega t} \, dt = \int_{M}^{L} e^{-i\omega t} \, dx(t)$$

$$= x(t)e^{-i\omega t} \bigg|_{M}^{L} - \int_{M}^{L} x(t) \, de^{-i\omega t}$$

$$= x(L)e^{-i\omega L} - x(M)e^{-i\omega M} + i\omega \int_{M}^{L} x(t)e^{-i\omega t} \, dt.$$

Now take limits $M \to -\infty$ and $L \to \infty.$
Differentiation in the time domain

**Example**

Find the Fourier transform of $x'(t)$ with $x(t) = e^{-\alpha t}u(t), \ \alpha > 0$.

- Use the product rule to differentiate $x(t)$:
  
  $x(t)' = e^{-\alpha t}\delta(t) - \alpha e^{-\alpha t}u(t)$
  
  $= e^{-\alpha t}\delta(t) - \alpha x(t)$
  
  $= \delta(t) - \alpha x(t)$.

- The Fourier transform of the right-hand side is
  
  $\mathcal{F}\{\delta(t) - \alpha x(t)\} = 1 - \alpha X(\omega) = 1 - \frac{\alpha}{\alpha + i\omega}$

  
  $= \frac{(\alpha + i\omega) - \alpha}{\alpha + i\omega} = \frac{i\omega}{\alpha + i\omega}$

  
  $= i\omega X(\omega)$.

The unit step-function

⚠️ **Be careful**

Let $U(\omega)$ be the Fourier transform of the step function $u(t)$, then using the differentiation theorem yields

\[ i\omega U(\omega) = i\omega \mathcal{F}\{u(t)\} = \mathcal{F}\{u'(t)\} = \mathcal{F}\{\delta(t)\} = 1. \]

This suggests $U(\omega) = \frac{1}{i\omega}$, but this is wrong!

- Notice that $\lim_{t \to \infty} u(t) \neq 0$: the step function does not satisfy the requirements of the differentiation theorem.
- Also, $\frac{1}{i\omega}$ is not defined for $\omega = 0$.

**Definition**

The Cauchy Principal Value of $1/x$ is defined as

\[ \text{CPV} \left( \frac{1}{x} \right) = \begin{cases} 
\frac{1}{x} & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases} \]
The unit step-function

- Write
  \[ U(\omega) = U(0) + \text{CPV} \frac{1}{i\omega}. \]
- Observe that \( u(t) + u(-t) = 1 \), hence
  \[
  2\pi\delta(\omega) = \mathcal{F}\{1\} = \mathcal{F}\{u(t)\} + \mathcal{F}\{u(-t)\} \\
  = U(\omega) + U(-\omega) \\
  = U(0) + \text{CPV} \frac{1}{i\omega} + U(0) - \text{CPV} \frac{1}{i\omega} \\
  = 2U(0).
  \]
- Hence \( U(0) = \pi\delta(\omega) \), and consequently

\[ u(t) \leftrightarrow \pi\delta(\omega) + \text{CPV} \frac{1}{i\omega}. \]  

Integration in the time domain

**Theorem**

Let \( x(t) \) be a piecewise continuous, integrable signal on \( \mathbb{R} \) with Fourier transform \( X(\omega) \). If \( X(0) = 0 \) then

\[
\mathcal{F}\left\{ \int_{-\infty}^{t} x(\tau) \, d\tau \right\} = \frac{1}{i\omega} X(\omega).
\]

- Note that \( \int_{-\infty}^{\infty} x(\tau) \, d\tau = \int_{-\infty}^{\infty} x(\tau)e^{-i0t} \, d\tau = X(0) \).
- Define \( y(t) = \int_{-\infty}^{t} x(\tau) \, d\tau \), then \( y(t) \) is piecewise smooth and \( \lim_{t \to \infty} y(t) = X(0) = 0 \).
- Since \( y'(t) = x(t) \) we have \( X(\omega) = i\omega Y(\omega) \), and then
  \[
  \mathcal{F}\left\{ \int_{-\infty}^{t} x(\tau) \, d\tau \right\} = \mathcal{F}\{y(t)\} \\
  = Y(\omega) = \frac{1}{i\omega} X(\omega).
  \]
Integration in the time domain

**Theorem**

Let \( x(t) \) be a piecewise continuous, integrable signal on \( \mathbb{R} \) with Fourier transform \( X(\omega) \). Then

\[
\mathcal{F} \left\{ \int_{-\infty}^{t} x(\tau) \, d\tau \right\} = \pi X(0) \delta(\omega) + \frac{1}{i\omega} X(\omega).
\]

- Note that \( \int_{-\infty}^{t} x(\tau) \, d\tau = (x \ast u)(t) \).
- Hence

\[
\mathcal{F} \left\{ \int_{-\infty}^{t} x(\tau) \, d\tau \right\} = X(\omega) U(\omega) = X(\omega) \left[ \pi \delta(\omega) + \frac{1}{i\omega} \right] = \pi X(0) \delta(\omega) + \frac{1}{i\omega} X(\omega).
\]

- See also example 4.3.10.

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Parseval's identity

**Theorem**

Let \( x(t) \) and \( y(t) \) be piecewise smooth signals with Fourier transforms \( X(\omega) \) and \( Y(\omega) \) respectively, then

\[
\int_{-\infty}^{\infty} x(t) y(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) Y(\omega) \, d\omega.
\]

**Corollary**

Let \( x(t) \) be a piecewise smooth signal with Fourier transforms \( X(\omega) \), then

\[
\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega.
\]

**Equation 4.3.11**
Parseval’s identity

\[ \int_{-\infty}^{\infty} X(\omega) Y(\omega) \, d\omega = \int_{-\infty}^{\infty} X(\omega) \int_{-\infty}^{\infty} y(t) e^{-i\omega t} \, dt \, d\omega \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\omega) y(t) e^{i\omega t} \, dt \, d\omega \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\omega) y(t) e^{i\omega t} \, d\omega \, dt \]

\[ = \int_{-\infty}^{\infty} y(t) \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} \, d\omega \, dt \]

\[ = \int_{-\infty}^{\infty} y(t) \cdot 2\pi x(t) \, dt \]

\[ = 2\pi \int_{-\infty}^{\infty} x(t) y(t) \, dt \]

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Frequency bands

**Definition**

- A **frequency band** is an interval in the frequency domain.
- The length of the interval is called the **bandwidth**.

**Definition**

A **energy of** \( x(t) \) **in frequency band** \( B \) is defined as

\[ \frac{1}{2\pi} \int_B |X(\omega)|^2 \, d\omega. \]

- Example: let \( \omega_0 > 0 \), then the energy of \( x(t) \) contained within the band \( |\omega| < \omega_0 \) is

\[ \Delta E = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} |X(\omega)|^2 \, d\omega. \]

- If \( x(t) \) is real, then \( |X(-\omega)| = |X(\omega)| = |X(\omega)| \), hence

\[ \Delta E = \frac{1}{\pi} \int_{0}^{\omega_0} |X(\omega)|^2 \, d\omega. \]
Frequency bands

Example 4.3.6

Let \( x(t) = e^{-t}u(t) \). Find the total energy \( E \) of \( x(t) \) and find the energy \( \Delta E \) in the band \( |\omega| < 4 \).

- See example 4.2.3: \( X(\omega) = \frac{1}{1 + i\omega} \).

\[
E = \frac{1}{\pi} \int_{0}^{\infty} |X(\omega)|^2 \, d\omega = \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{1 + \omega^2} \, d\omega
\]

\[
= \frac{1}{\pi} \arctan(\omega) \bigg|_{0}^{\infty} = \frac{1}{\pi} \left( \frac{\pi}{2} - 0 \right) = .5
\]

- A similar calculation shows

\[
\Delta E = \frac{1}{\pi} \int_{0}^{4} |X(\omega)|^2 \, d\omega = \frac{1}{\pi} \arctan(4) \approx 0.422021
\]

- Hence the fraction of energy in the band \( |\omega| < 4 \) is

\[
\frac{\Delta E}{E} \times 100 \approx 84.4\%.
\]

Convolution in the frequency domain

Theorem – Modulation property

Let \( x(t) \) and \( y(t) \) be piecewise smooth, absolutely integrable and with finite energy. Assume that the Fourier transforms are \( X(\omega) \) and \( Y(\omega) \) respectively. Then the \( F\{x(t) \, y(t)\} \) exists and

\[
F\{x(t) \, y(t)\} = \frac{1}{2\pi} X(\omega) \ast Y(\omega)
\]

The output of a multiplier is the product of the input signals which transform to a (scaled) convolution in the frequency domain.
Shift in the frequency domain

Multiply input \( x(t) \) with a time-harmonic signal \( e^{j\omega_0 t} \).

The Fourier transform of the output is
\[
\frac{1}{2\pi} X(\omega) \ast (2\pi \delta(\omega - \omega_0)) = X(\omega) \ast \delta(\omega - \omega_0) \\
= \int_{-\infty}^{\infty} X(\sigma) \delta((\omega - \sigma) - \omega_0) \, d\sigma = X(\omega - \omega_0).
\]

**Theorem**

Let \( x(t) \) be a signal with Fourier transform \( X(\omega) \), then
\[
x(t) e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)
\]

Modulation

**Definition**

Modulation is the process of superimposing a signal to a carrier signal.

- Data signal
  - Carrier signal
  - Amplitude modulation
  - Frequency modulation
In amplitude modulation a data signal $x(t)$ is multiplied with a carrier signal $m(t) = \cos \omega_0 t$.

The modulated signal $y(t) = x(t)m(t)$ has Fourier transform

$$Y(\omega) = \frac{1}{2\pi} X(\omega) \ast \pi \left( \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right)$$

$$= \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0).$$

Demodulate signal $y(t)$ by multiplying it with the carrier signal again. If $z(t) = y(t)m(t)$ then:

$$Z(\omega) = \frac{1}{2} Y(\omega - \omega_0) + \frac{1}{2} Y(\omega + \omega_0)$$

$$= \frac{1}{4} X(\omega - 2\omega_0) + \frac{1}{4} X(\omega) + \frac{1}{4} X(\omega + 2\omega_0).$$

Observe that

$$\cos^2 \omega_0 t = \frac{1}{2} + \frac{1}{2} \cos(2\omega_0 t)$$

$$\leftrightarrow \frac{1}{2} \delta(\omega) + \frac{1}{4} \delta(\omega - 2\omega_0) + \frac{1}{4} \delta(\omega + 2\omega_0).$$

The original signal can be retrieved by processing $z(t)$ through a low pass filter.
The Dirac comb

Definition

- **The Dirac comb** with period \( T \) is a pulse train consisting of Dirac pulses, and is defined as
  \[
  \mathcal{I}_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).
  \]

- The Dirac comb with period 1 is denoted as \( \mathcal{I}(t) \).

- The symbol \( \mathcal{I} \) is the Cyrillic character “Sha”.
- The Dirac comb is therefore sometimes called the **Shah function**.

Example 4.2.10

The Dirac comb is periodic with period \( T \) and (with \( \omega_0 = 2\pi/T \)) has Fourier coefficients

\[
 c_n = \frac{1}{T} \int_{-T/2}^{T/2} \mathcal{I}_T(t) e^{-i\omega_0 nt} \, dt
 = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-i\omega_0 nt} \, dt = \frac{1}{T} e^{-i\omega_0 n \cdot 0} = \frac{1}{T}.
\]

- The Dirac comb has Fourier series expansion
  \[
  \mathcal{I}_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{in\omega_0 t}.
  \]
- The Dirac comb has Fourier transform
  \[
  \mathcal{I}_T(t) \leftrightarrow \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)
 = \sum_{n=-\infty}^{\infty} \delta(\omega/\omega_0 - n) = \mathcal{I}(\omega/\omega_0).
  \]

- The Fourier transform of a Dirac comb is a Dirac comb.
The Dirac comb

- Observe that \( \Pi_T(t) \) does not satisfy the conditions of the Fundamental Theorem of Fourier Series.

**Question**

Why is \( \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{i\omega_0 t} \) ? \((*)\)

- Let \( T = 1 \), then equation \((*)\) boils down to
  \[
  \sum_{n=-\infty}^{\infty} \delta(t - n) = \sum_{n=-\infty}^{\infty} e^{2\pi i nt}.
  \]

- If \( t = k \in \mathbb{Z} \): \( \sum_{n=-\infty}^{\infty} \delta(k - n) = \infty \) and \( \sum_{n=-\infty}^{\infty} e^{2\pi i nk} = \infty \).

- If \( t \notin \mathbb{Z} \), then \( \sum_{n=-\infty}^{\infty} \delta(t - n) = 0 \), but why \( \sum_{n=-\infty}^{\infty} e^{2\pi i nt} = 0 ? \)

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The Dirac comb

- \( \sum_{n=-\infty}^{\infty} e^{2\pi i nt} = 0, \quad t \notin \mathbb{Z} \). \((**)\)

Equation \((***)\) can only be understood with **distributions**.

- Define \( \zeta = e^{2\pi it} \), then \( |\zeta| = 1 \), \( \zeta \neq 1 \), and \((***)\) becomes
  \[
  \sum_{n=-\infty}^{\infty} \zeta^n = 0.
  \]

- If \( t \) is rational, then \( \zeta \) has finite order: \( \zeta^N = 1 \) for \( N > 0 \).
  \[
  \sum_{n=-\infty}^{\infty} \zeta^n = \ldots + 1 + \zeta + \zeta^2 + \ldots + \zeta^N - 1 + \ldots \\
  = \ldots + \frac{\zeta^N - 1}{\zeta - 1} + \ldots = \ldots + 0 + \ldots .
  \]

- If \( t \) is irrational, then all powers \( \zeta^n \) fill the unit circle, hence
  \[
  \sum_{n=-\infty}^{\infty} \zeta^n \approx \int_{0}^{2\pi} e^{i\tau} \, d\tau = \frac{1}{i} \left( e^{2\pi i} - 1 \right) = 0.
  \]
Sampling of a signal $x(t)$ is the process of obtaining values of $x(t)$ at regular time intervals.

The sampled signal $x_s(t)$ is obtained by multiplying $x(t)$ with a Dirac comb.

Using the sample property of $\delta(t)$ we see

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT).$$

The Fourier transform of sampled signals

- Example 4.3.13: the Fourier transform of $III(t)$ is

$$III(\omega/\omega_0) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0).$$

- The Fourier transform of the sampled signal $x_s(t)$ is

$$\frac{1}{2\pi} X(\omega) \ast III(\omega/\omega_0) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega) \ast \delta(\omega - n\omega_0)$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_0).$$

- The Fourier transform of $x_s(t)$ is obtained by adding shifted and scaled copies of $X(\omega)$.
The sampling theorem

If the copies of $X(\omega)$ overlap, then the transform $X(\omega)$ cannot be obtained from $X_s(\omega)$ by filtering.

This phenomenon is called aliasing.

If the bandwidth of $X(\omega)$ is $\omega_B$, and $\omega_0 > 2\omega_B$, then $X(\omega)$ can be retrieved from $X_s(\omega)$ by filtering.

The frequency $2\omega_B$ is called the Nyquist frequency.